

# THE BLOCK STRUCTURE OF THE PARTITION ALGEBRAS IN CHARACTERISTIC $p \geq 0$

C. BOWMAN, M. DE VISSCHER, AND O. KING

ABSTRACT. In this paper we describe the blocks of the partition algebra over a field of positive characteristic.

## INTRODUCTION

The partition algebra was originally defined by Martin in [Mar94] over the complex field  $\mathbb{C}$  as a generalisation of the Temperley-Lieb algebra for  $\delta$ -state  $n$ -site Potts models in statistical mechanics. Although this interpretation requires  $\delta$  to be integral, it is possible to define the partition algebra  $P_n^{\mathbb{F}}(\delta)$  over any field  $\mathbb{F}$  and for any  $\delta \in \mathbb{F}$ . It was shown in [Xi99] that the partition algebra  $P_n^{\mathbb{F}}(\delta)$  over an arbitrary field  $\mathbb{F}$  is a cellular algebra, with cell modules indexed by partitions  $\lambda$  of size at most  $n$ . If  $\text{char } \mathbb{F} = 0$  and  $\delta \neq 0$ , these partitions also label a complete set of non-isomorphic simple modules, given by the heads of the corresponding cell modules. Moreover, Martin gave a complete combinatorial description of how the cell modules decompose into simple modules by introducing the notion of  $\delta$ -pairs of partitions.

If  $\text{char } \mathbb{F} = p > 0$  and  $\delta \neq 0$  then the simple modules are indexed by the subset of  $p$ -regular partitions. But very little investigation into the structure of the cell modules has been made in this case. The first natural problem to consider is the determination of the block structure of the algebra. This paper aims to solve this problem completely.

Sections 1,2 and 3 contain all the background needed for this paper. Section 4 starts with the description, in terms of  $\delta$ -pairs, of the representation theory of the partition algebra over a field of characteristic zero due to P.Martin. We then reformulate his results in terms of an action of the Weyl group  $W$  of type  $A$  on the set of partition. Section 5 contains the main result of this paper, namely a description of the blocks of the partition algebra over a field of positive characteristic in terms of the action of the corresponding affine Weyl group of type  $A$ .

**Notations:** Throughout the paper, we fix a prime number  $p > 2$  and a  $p$ -modular system  $(K, R, k)$ , that is,  $R$  is a discrete valuation ring with maximal ideal  $\mathfrak{m} = (\pi)$ ,  $K = \text{Frac}(R)$  is its field of fractions (of characteristic zero) and  $k = R/\mathfrak{m}$  is the residue field of characteristic  $p > 2$ . We assume that  $K$  and  $k$  are algebraically closed. We also fix  $\delta \in R$  and assume that its projection in  $k$  is non-zero. We use the same notation for  $\delta \in R$  and its projection in  $k$ . We will also use  $\mathbb{F}$  to denote either  $K$  or  $k$ .

## 1. COMBINATORICS OF PARTITIONS

**1.1. Partitions and Young diagrams.** Given a natural number  $n$ , we define a partition  $\lambda = (\lambda_1, \lambda_2, \dots)$  of  $n$  to be a weakly decreasing sequence of non-negative integers such that  $\sum_{i \geq 0} \lambda_i = n$ . As we have  $\lambda_i = 0$  for  $i \gg 0$  we will often truncate the sequence and write  $\lambda = (\lambda_1, \dots, \lambda_l)$ , where

---

*Date:* March 7, 2014.

*2000 Mathematics Subject Classification.* 20C30.

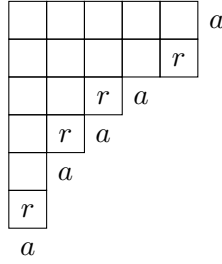


FIGURE 1. The Young diagram of  $\lambda = (5^2, 3, 2, 1^2)$ . Removable nodes are marked by  $r$  and addable nodes by  $a$ .

$\lambda_l \neq 0$  and  $\lambda_{l+1} = 0$ . We also combine repeated entries and use exponents, for instance the partition  $(5, 5, 3, 2, 1, 1, 0, 0, 0, \dots)$  of 17 will be written  $(5^2, 3, 2, 1^2)$ . We use the notation  $\lambda \vdash n$  to mean  $\lambda$  is a partition of  $n$ . We call  $n$  the degree of the partition and write  $n = |\lambda|$ .

We say that a partition  $\lambda = (\lambda_1, \dots, \lambda_l)$  is  $p$ -singular if there exists  $t$  such that

$$\lambda_t = \lambda_{t+1} = \dots = \lambda_{t+p-1} > 0$$

i.e. some (non-zero) part of  $\lambda$  is repeated  $p$  or more times. Partitions that are not  $p$ -singular we call  $p$ -regular.

We denote by  $\Lambda_n$  the set of all partitions of  $n$ . We also define  $\Lambda_{\leq n} = \cup_{0 \leq i \leq n} \Lambda_i$ . We will also consider  $\Lambda_n^*$  and  $\Lambda_{\leq n}^*$  the subsets of  $p$ -regular partitions of  $\Lambda_n$  and  $\Lambda_{\leq n}$  respectively.

There is a partial order on  $\Lambda_{\leq n}$  called the *dominance order with size*, which we denote by  $<_d$ . For  $\lambda, \mu \in \Lambda_{\leq n}$  we say that  $\lambda <_d \mu$  if either  $|\lambda| < |\mu|$ , or  $\lambda \neq \mu$ ,  $|\lambda| = |\mu|$  and  $\sum_{i=1}^j \lambda_i \leq \sum_{i=1}^j \mu_i$  for all  $j \geq 1$ . We write  $\lambda \leq_d \mu$  to mean  $\lambda <_d \mu$  or  $\lambda = \mu$ .

To each partition  $\lambda$  we may associate the Young diagram

$$[\lambda] = \{(x, y) \mid x, y \in \mathbb{Z}, 1 \leq x \leq l, 1 \leq y \leq \lambda_x\}$$

An element  $(x, y)$  of  $[\lambda]$  is called a *node*. If  $\lambda_{i+1} < \lambda_i$ , then the node  $(i, \lambda_i)$  is called a *removable* node of  $\lambda$ . If  $\lambda_{i-1} > \lambda_i$ , then we say the node  $(i, \lambda_i + 1)$  of  $[\lambda] \cup \{(i, \lambda_i + 1)\}$  is an *addable* node of  $\lambda$ . This is illustrated in Figure 1. If a partition  $\mu$  is obtained from  $\lambda$  by removing a removable (resp. adding an addable) node then we write  $\mu \triangleleft \lambda$  (resp.  $\mu \triangleright \lambda$ ).

Each node  $\epsilon = (x, y)$  of  $[\lambda]$  has an associated integer,  $c(\epsilon)$ , called the *content* of  $\epsilon$ , given by  $c(\epsilon) = y - x$ . We write

$$\text{ct}(\lambda) = \sum_{\epsilon \in [\lambda]} c(\epsilon) \tag{1.1}$$

**1.2. Abacus.** Following [JK81, Section 2.7] we can associate to each partition and prime number  $p$  an abacus diagram, consisting of  $p$  columns, known as runners, and a configuration of beads across these. By convention we label the runners from left to right, starting with 0, and the positions on the abacus are also numbered from left to right, working down from the top row, starting with 0 (see Figure 2). Given a partition  $\lambda = (\lambda_1, \dots, \lambda_l) \vdash n$ , fix a positive integer  $b \geq n$  and construct the  $\beta$ -sequence of  $\lambda$ , defined to be

$$\beta(\lambda, b) = (\lambda_1 - 1 + b, \lambda_2 - 2 + b, \dots, \lambda_l - l + b, -(l+1) + b, \dots, 2, 1, 0)$$

Then place a bead on the abacus in each position given by  $\beta(\lambda, b)$ , so that there are a total of  $b$  beads across the runners. Note that for a fixed value of  $b$ , the abacus is uniquely determined by  $\lambda$ , and any such abacus arrangement corresponds to a partition simply by reversing the above. Here is an example of such a construction.

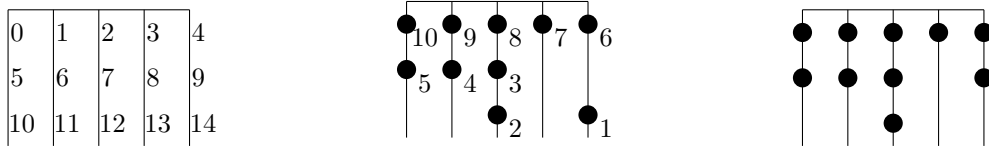


FIGURE 2. The positions on the abacus with 5 runners, the arrangement of beads (numbered) representing  $\lambda = (5, 4)$ , and the corresponding 5-core

*Example 1.1.* In this example we will fix the values  $p = 5, n = 9, b = 10$  and represent the partition  $\lambda = (5, 4)$  on the abacus. Following the above process, we first calculate the  $\beta$ -sequence of  $\lambda$ :

$$\begin{aligned} \beta(\lambda, 10) &= (5 - 1 + 10, 4 - 2 + 10, -3 + 10, -4 + 10, \dots, -9 + 10, -10 + 10) \\ &= (14, 12, 7, 6, 5, 4, 3, 2, 1, 0) \end{aligned}$$

The next step is to place beads on the abacus in the corresponding positions. We also number the beads, so that bead 1 occupies position  $\lambda_1 - 1 + b$ , bead 2 occupies position  $\lambda_2 - 2 + b$  and so on. The labelled spaces and the final abacus with labelled beads are shown in Figure 2.

After fixing values of  $p$  and  $b$ , we will abuse notation and write  $\lambda$  for both the partition and the corresponding abacus with  $p$  runners and  $b$  beads. We may also then define  $\Gamma(\lambda, b) = (\Gamma(\lambda, b)_0, \Gamma(\lambda, b)_1, \dots, \Gamma(\lambda, b)_{p-1})$ , where

$$\Gamma(\lambda, b)_i = |\{j : \beta(\lambda, b)_j \equiv i \pmod{p}\}| \quad (1.2)$$

so that  $\Gamma(\lambda, b)$  records the number of beads on each runner of the abacus of  $\lambda$ .

We define the  $p$ -core of the partition  $\lambda$  to be the partition  $\mu$  whose abacus is obtained from that of  $\lambda$  by sliding all the beads as far up their runners as possible. In particular, we have  $\Gamma(\lambda, b) = \Gamma(\mu, b)$ . It can be shown that the  $p$ -core  $\mu$  is independent of the choice of  $b$ , and so depends only on  $\lambda$  and  $p$ . The 5-core of the partition  $(5, 4)$  given in the example above is the partition  $(3, 1)$ . Its abacus is illustrated in Figure 2.

## 2. REPRESENTATION THEORY OF THE SYMMETRIC GROUP

The group algebra  $R\mathfrak{S}_n$  is a cellular algebra, as shown in [GL96]. The cell modules are indexed by partitions  $\lambda = (\lambda_1, \dots, \lambda_l)$  of  $n$ , and are more commonly known as *Specht modules*. We denote the Specht module indexed by  $\lambda$  by  $S_R^\lambda$ . These can be constructed explicitly, see for example [Jam78, Chapter 4]. We define the  $K\mathfrak{S}_n$ -module  $S_K^\lambda := K \otimes_R S_R^\lambda$  and the  $k\mathfrak{S}_n$ -module  $S_k^\lambda := k \otimes_R S_R^\lambda$ .

**Theorem 2.1** ([Jam78, Theorem 4.12]). *The set of all  $S_K^\lambda$ ,  $\lambda \in \Lambda_n$ , gives a complete set of pairwise non-isomorphic simple  $K\mathfrak{S}_n$ -modules.*

**Theorem 2.2** ([Jam78, Theorem 11.5]). *For  $\lambda \in \Lambda_n^*$ , the Specht module  $S_k^\lambda$  has simple head, denoted by  $D_k^\lambda$ . Moreover the set of all  $D_k^\lambda$ ,  $\lambda \in \Lambda_n^*$  gives a complete set of pairwise non-isomorphic simple  $k\mathfrak{S}_n$ -modules.*

The problem of describing all composition factors of the Specht modules for  $k\mathfrak{S}_n$  remains wide open. But the blocks of this algebra are well-known.

**Theorem 2.3** (Nakayama's Conjecture). [JK81, Chapter 6] *Two partitions  $\lambda, \mu \in \Lambda_n$  label Specht modules in the same block for  $k\mathfrak{S}_n$  if and only if they have the same  $p$ -core, that is  $\Gamma(\lambda, b) = \Gamma(\mu, b)$  for some (and hence all)  $b \geq n$ .*

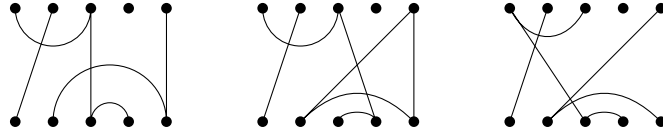
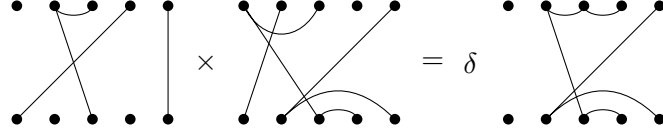
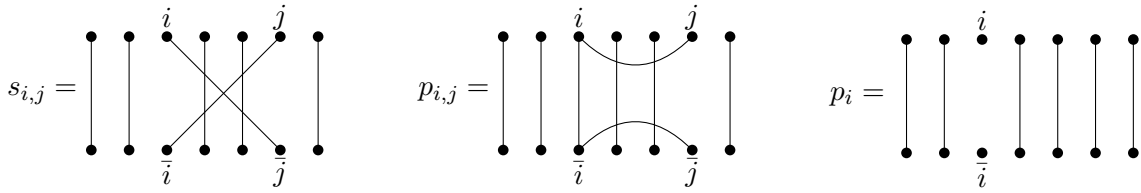
FIGURE 3. Three diagrams representing the set-partition  $\{\{1, 3, \bar{3}, \bar{4}\}, \{2, \bar{1}\}, \{4\}, \{5, \bar{2}, \bar{5}\}\}$ FIGURE 4. Multiplication of two diagrams in  $P_5^R(\delta)$ .

FIGURE 5. Generators of the partition algebra

### 3. THE PARTITION ALGEBRA: DEFINITION AND CELLULARITY

**3.1. Definitions and first properties.** For a fixed  $n \in \mathbb{N}$  and  $\delta \in R$ , we define the partition algebra  $P_n^R(\delta)$  to be the set of linear combinations of set-partitions of  $\{1, 2, \dots, n, \bar{1}, \bar{2}, \dots, \bar{n}\}$ . We call each part of a set-partition a *block*. For instance,

$$\{\{1, 3, \bar{3}, \bar{4}\}, \{2, \bar{1}\}, \{4\}, \{5, \bar{2}, \bar{5}\}\}$$

is a set-partition with  $n = 5$  consisting of 4 blocks. Any block with  $\{i, \bar{j}\}$  as a subset for some  $i$  and  $j$  is called a *propagating block*.

We can represent each set-partition by an *partition diagram*, consisting of two rows of  $n$  nodes with arcs between nodes in the same block.

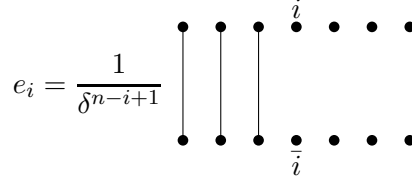
Note that in general there are many diagrams corresponding to the same set-partition. For example if we take  $n = 5$ , then some of the diagrams representing the set-partition  $\{\{1, 3, \bar{3}, \bar{4}\}, \{2, \bar{1}\}, \{4\}, \{5, \bar{2}, \bar{5}\}\}$  are given in Figure 3. We will identify diagrams corresponding to the same set partition.

Multiplication in the partition algebra is by concatenation of diagrams in the following way: to obtain the result  $x \cdot y$  given diagrams  $x$  and  $y$ , place  $x$  on top of  $y$  and identify the bottom nodes of  $x$  with those on top of  $y$ . This new diagram may contain a number,  $t$  say, of blocks in the centre not connected to the northern or southern edges of the diagram. These we remove and multiply the final result by  $\delta^t$ . An example is given in Figure 4.

It is easy to see that the elements  $s_{i,j}$ ,  $p_{i,j}$  ( $1 \leq i < j \leq n$ ) and  $p_i$  ( $1 \leq i \leq n$ ) defined in Figure 5 generate  $P_n^R(\delta)$ . We write  $s_i := s_{i,i+1}$  and  $p_{i+\frac{1}{2}} := p_{i,n}$  for  $1 \leq i \leq n-1$ .

Notice that multiplication in  $P_n^R(\delta)$  cannot increase the number of propagating blocks. We therefore have a filtration of  $P_n^R(\delta)$  by the number of propagating blocks.

In what follows we will work over the field  $\mathbb{F} = K$  or  $k$  and assume that  $\delta \in \mathbb{F}$  is non-zero. This allows us to realise the filtration by use of the idempotents  $e_i$  defined in Figure 6. So we have

FIGURE 6. The idempotent  $e_i$ 

$$J_n^{(0)} \subset J_n^{(1)} \subset \dots J_n^{(n-1)} \subset J_n^{(n)} = P_n^{\mathbb{F}}(\delta) \quad (3.1)$$

where  $J_n^{(r)} = P_n^{\mathbb{F}}(\delta)e_{r+1}P_n^{\mathbb{F}}(\delta)$  is spanned by all diagrams with at most  $r$  propagating blocks.

We also use  $e_i$  to construct algebra isomorphisms

$$\Phi_n : P_{n-1}^{\mathbb{F}}(\delta) \longrightarrow e_n P_n^{\mathbb{F}}(\delta) e_n \quad (3.2)$$

taking a diagram in  $P_{n-1}^{\mathbb{F}}(\delta)$  and adding an extra northern and southern node to the right hand end. Using this and following [Gre80] we obtain an exact localisation functor

$$\begin{aligned} F_n : P_n^{\mathbb{F}}(\delta)\text{-}\mathbf{mod} &\longrightarrow P_{n-1}^{\mathbb{F}}(\delta)\text{-}\mathbf{mod} \\ M &\longmapsto e_n M \end{aligned} \quad (3.3)$$

and a right exact globalisation functor

$$\begin{aligned} G_n : P_n^{\mathbb{F}}(\delta)\text{-}\mathbf{mod} &\longrightarrow P_{n+1}^{\mathbb{F}}(\delta)\text{-}\mathbf{mod} \\ M &\longmapsto P_{n+1}^{\mathbb{F}}(\delta)e_{n+1} \otimes_{P_n^{\mathbb{F}}(\delta)} M \end{aligned} \quad (3.4)$$

Since  $F_{n+1}G_n(M) \cong M$  for all  $M \in P_n^{\mathbb{F}}(\delta)\text{-}\mathbf{mod}$ ,  $G_n$  is a full embedding of categories. From the filtration (3.1) we see that

$$P_n^{\mathbb{F}}(\delta)/J_n^{(n-1)} \cong \mathbb{F}\mathfrak{S}_n \quad (3.5)$$

and so using (3.2) and following [Gre80], we see that the simple  $P_n^{\mathbb{F}}(\delta)$ -modules are indexed by the set  $\Lambda_{\leq n}$  if  $\mathbb{F} = K$  and by the set  $\Lambda_{\leq n}^*$  if  $\mathbb{F} = k$ .

We will also need to consider the algebra  $P_{n-\frac{1}{2}}^{\mathbb{F}}(\delta)$ , which is the subalgebra of  $P_n^{\mathbb{F}}(\delta)$  spanned by all set-partitions with  $n$  and  $\bar{n}$  in the same block. As in (3.1) we have a filtration of this algebra defined by the number of propagating blocks:

$$J_{n-\frac{1}{2}}^{(1)} \subset J_{n-\frac{1}{2}}^{(2)} \subset \dots J_{n-\frac{1}{2}}^{(n-1)} \subset J_{n-\frac{1}{2}}^{(n)} = P_{n-\frac{1}{2}}^{\mathbb{F}}(\delta) \quad (3.6)$$

where  $J_{n-\frac{1}{2}}^{(r)}$  is spanned by all diagrams with at most  $r$  propagating blocks. Note that since we require the nodes  $n$  and  $\bar{n}$  to be in the same block, we always have at least one propagating block. Also since  $n$  and  $\bar{n}$  must always be joined, we see that  $P_{n-\frac{1}{2}}^{\mathbb{F}}/J_{n-\frac{1}{2}}^{(n-1)} \cong \mathbb{F}\mathfrak{S}_{n-1}$ , and so following the argument for  $P_n^{\mathbb{F}}(\delta)$  above we see that the simple  $P_{n-\frac{1}{2}}^{\mathbb{F}}(\delta)$ -modules are indexed by  $\Lambda_{\leq n-1}$  of  $\mathbb{F} = K$  and by  $\Lambda_{\leq n-1}^*$  of  $\mathbb{F} = k$ .

Note that inclusion gives us an injective algebra homomorphism

$$\begin{aligned} P_n^R(\delta) &\longrightarrow P_{n+\frac{1}{2}}^R(\delta) \\ d &\longmapsto d \cup \{ \{n+1, \overline{n+1}\} \} \end{aligned}$$

This allows us to define restriction and induction functors

$$\text{res}_n : P_n^{\mathbb{F}}(\delta)\text{-}\mathbf{mod} \longrightarrow P_{n-\frac{1}{2}}^{\mathbb{F}}(\delta)\text{-}\mathbf{mod}$$

$$\begin{aligned}
M &\longmapsto M|_{P_{n-\frac{1}{2}}^{\mathbb{F}}(\delta)} \\
\text{ind}_n : P_n^{\mathbb{F}}(\delta)\text{-}\mathbf{mod} &\longrightarrow P_{n+\frac{1}{2}}^{\mathbb{F}}(\delta)\text{-}\mathbf{mod} \\
M &\longmapsto P_{n+\frac{1}{2}}^{\mathbb{F}}(\delta) \otimes_{P_n^{\mathbb{F}}(\delta)} M
\end{aligned} \tag{3.7}$$

**3.2. Cellularity of  $P_n^{\mathbb{F}}(\delta)$ .** It was shown in [Xi99] that the partition algebra is cellular as defined in [GL96]. The cell modules  $\Delta_{\lambda}^{\mathbb{F}}(n; \delta)$  are indexed by the set of partitions  $\lambda \in \Lambda_{\leq n}$ , and the cellular ordering is given by the reverse of  $<_d$ . When  $\lambda \vdash n$ , we obtain  $\Delta_{\lambda}^{\mathbb{F}}(n; \delta)$  by lifting the Specht module  $S_{\mathbb{F}}^{\lambda}$  to the partition algebra using (3.5). When  $\lambda \vdash n - t$  for some  $t > 0$ , we obtain the cell module by

$$\Delta_{\lambda}^{\mathbb{F}}(n; \delta) = G_{n-1}G_{n-2} \dots G_{n-t}\Delta_{\lambda}^{\mathbb{F}}(n-t; \delta)$$

Over  $K$ , each of the cell modules has a simple head  $L_{\lambda}^K(n; \delta)$ , and these form a complete set of non-isomorphic simple  $P_n^K(\delta)$ -modules. Over  $k$ , the heads  $L_{\lambda}^k(n; \delta)$  of cell modules labelled by  $p$ -regular partitions  $\lambda \in \Lambda_{\leq n}^*$  provide a complete set of non-isomorphic simple  $P_n^k(\delta)$ -modules. Moreover we have that  $[\Delta_{\lambda}^k(n, \delta) : L_{\mu}^k(n, \delta)] \neq 0$  implies  $\mu \geq_d \lambda$ .

When the context is clear, we will write  $\Delta_{\lambda}^{\mathbb{F}}(n)$  and  $L_{\lambda}^{\mathbb{F}}(n)$  to mean  $\Delta_{\lambda}^{\mathbb{F}}(n; \delta)$  and  $L_{\lambda}^{\mathbb{F}}(n; \delta)$  respectively.

By definition we have that the localisation and globalisation functors preserve the cell modules. More precisely we have

$$\begin{aligned}
F_n(\Delta_{\lambda}^{\mathbb{F}}(n)) &\cong \begin{cases} \Delta_{\lambda}^{\mathbb{F}}(n-1) & \text{if } \lambda \in \Lambda_{\leq n-1} \\ 0 & \text{otherwise} \end{cases} \\
G_n(\Delta_{\lambda}^{\mathbb{F}}(n)) &\cong \Delta_{\lambda}^{\mathbb{F}}(n+1)
\end{aligned}$$

We also have an explicit construction of the cell modules over  $R$ . Let  $I(n, t)$  be the set of partition diagrams with precisely  $t$  propagating blocks and  $\overline{t+1}, \overline{t+2}, \dots, \overline{n}$  each in singleton blocks. Then denote by  $V(n, t)$  the free  $R$ -module with basis  $I(n, t)$ . For a partition  $\lambda \vdash t$  we can easily show that  $\Delta_{\lambda}^R(n) \cong V(n, t) \otimes_{\mathfrak{S}_t} S_R^{\lambda}$ , where  $S_R^{\lambda}$  is the Specht module and the right action of  $\mathfrak{S}_t$  on  $V(n, t)$  is by permutation of the  $t$  left most southern nodes. The action of  $P_n^R(\delta)$  on  $\Delta_{\lambda}^R(n)$  is as follows: given a partition diagram  $x \in P_n^R(\delta)$ ,  $v \in I(n, t)$  and  $s \in S_R^{\lambda}$ , we define the element

$$x(v \otimes s) = (xv) \otimes s$$

where  $(xv)$  is the product of the partition diagrams if the concatenation of  $x$  and  $v$  has  $t$  propagating blocks, and is 0 otherwise. (Note that when  $\lambda \vdash n$  we have  $\Delta_{\lambda}^R(n) = S_{\lambda}^R$ ).

The cell modules over  $K$  and  $k$  are then obtained as

$$\Delta_{\lambda}^K(n) = K \otimes_R \Delta_{\lambda}^R(n) \text{ and } \Delta_{\lambda}^k(n) = k \otimes_R \Delta_{\lambda}^R(n)$$

The algebra  $P_{n-\frac{1}{2}}^{\mathbb{F}}(\delta)$  is also cellular [Mar00]. We can construct the cell modules in a similar way. Let  $I(n - \frac{1}{2}, t)$  be the set of partition diagrams with precisely  $t$  propagating blocks, one of which contains  $n$  and  $\bar{n}$ , with  $\overline{t+1}, \overline{t+2}, \dots, \overline{n-1}$  each in singleton blocks. Then denote by  $V(n - \frac{1}{2}, t)$  the free  $R$ -module with basis  $I(n - \frac{1}{2}, t)$ . For a partition  $\lambda \vdash t - 1$  we can define  $\Delta_{\lambda}^R(n - \frac{1}{2}) \cong V(n - \frac{1}{2}, t) \otimes_{\mathfrak{S}_{t-1}} S_R^{\lambda}$ , where  $S_R^{\lambda}$  is a Specht module and the right action of  $\mathfrak{S}_{t-1}$  on  $V(n - \frac{1}{2}, t)$  is by permuting the  $t - 1$  left most southern nodes. The action of  $P_{n-\frac{1}{2}}^R(\delta)$  is the same as in the previous case.

The following branching rules were given in [Mar00, Proposition 7].

$$\begin{aligned}
0 &\longrightarrow \Delta_\lambda^{\mathbb{F}}(n) \longrightarrow \text{res}_{n+\frac{1}{2}} \Delta_\lambda^{\mathbb{F}}(n + \tfrac{1}{2}) \longrightarrow \bigoplus_{\mu \triangleright \lambda} \Delta_\mu^{\mathbb{F}}(n) \longrightarrow 0 \\
0 &\longrightarrow \bigoplus_{\mu \triangleleft \lambda} \Delta_\mu^{\mathbb{F}}(n - \tfrac{1}{2}) \longrightarrow \text{res}_n \Delta_\lambda^{\mathbb{F}}(n) \longrightarrow \Delta_\lambda^{\mathbb{F}}(n - \tfrac{1}{2}) \longrightarrow 0 \\
0 &\longrightarrow \Delta_\lambda^{\mathbb{F}}(n) \longrightarrow \text{ind}_{n-\frac{1}{2}} \Delta_\lambda^{\mathbb{F}}(n - \tfrac{1}{2}) \longrightarrow \bigoplus_{\mu \triangleright \lambda} \Delta_\mu^{\mathbb{F}}(n) \longrightarrow 0 \\
0 &\longrightarrow \bigoplus_{\mu \triangleleft \lambda} \Delta_\mu^{\mathbb{F}}(n + \tfrac{1}{2}) \longrightarrow \text{ind}_n \Delta_\lambda^{\mathbb{F}}(n) \longrightarrow \Delta_\lambda^{\mathbb{F}}(n + \tfrac{1}{2}) \longrightarrow 0
\end{aligned} \tag{3.8}$$

Moreover, Martin proved the existence of a Morita equivalence between  $P_{n+\frac{1}{2}}^{\mathbb{F}}(\delta)$  and  $P_n^{\mathbb{F}}(\delta - 1)$ .

**Proposition 3.1.** [Mar00, Section 3] *Define the idempotent*

$$\xi_{n+1} = \prod_{i=1}^n (1 - p_{i,n+1}) \in P_{n+1}^{\mathbb{F}}(\delta)$$

(i) *We have an algebra isomorphism*

$$\xi_{n+1} P_{n+\frac{1}{2}}^{\mathbb{F}}(\delta) \xi_{n+1} \cong P_n^{\mathbb{F}}(\delta - 1)$$

*which induces a Morita equivalence between the categories  $P_{n+\frac{1}{2}}^{\mathbb{F}}(\delta)\text{-mod}$  and  $P_n^{\mathbb{F}}(\delta - 1)\text{-mod}$ . More precisely, using the above isomorphism the functors*

$$\Phi : P_{n+\frac{1}{2}}^{\mathbb{F}}(\delta)\text{-mod} \longrightarrow P_n^{\mathbb{F}}(\delta - 1)\text{-mod}$$

$$M \longmapsto \xi_{n+1} M$$

$$\text{and } \Psi : P_n^{\mathbb{F}}(\delta - 1)\text{-mod} \longrightarrow P_{n+\frac{1}{2}}^{\mathbb{F}}(\delta)\text{-mod}$$

$$N \longmapsto P_{n+\frac{1}{2}}^{\mathbb{F}}(\delta) \xi_{n+1} \otimes_{P_n^{\mathbb{F}}(\delta - 1)} N$$

*define an equivalence of categories.*

(ii) *This equivalence preserves the cellular structure of these algebras and we have*

$$\Phi(\Delta_\lambda^{\mathbb{F}}(n + \tfrac{1}{2})) \cong \Delta_\lambda^{\mathbb{F}}(n)$$

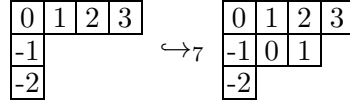
*for all  $\lambda \in \Lambda_{\leq n}$ .*

*Proof.* The proof of (i) is given in [Mar00, Section 3]. The argument for part (ii) is similar to those given in [Mar00]. Simply note that for any diagram  $v \in I(n + \frac{1}{2}, t)$  we have  $\xi_{n+1} v = 0$  unless  $\{n + 1, \overline{n + 1}\}$  is a block of  $v$  and so we have  $\xi_{n+1} V(n + \frac{1}{2}, t) \cong V(n, t - 1)$ .  $\square$

#### 4. ORDINARY REPRESENTATION THEORY OF THE PARTITION ALGEBRA

In this section we recall the results due to P. Martin [Mar96] on the representation theory of the partition algebra over a field of characteristic zero, and then reinterpret these in a geometrical setting.

The partition algebra  $P_n^K(\delta)$  is semisimple unless  $\delta \in \mathbb{Z}$ , see [Mar96] for a precise semisimplicity criterion. In general, the blocks of this algebra can be described completely. Before recalling this result, we introduce some notation.

FIGURE 7. An example of a  $\delta$ -pair when  $\delta = 7$ 

*Definition 4.1.* Let  $\lambda \in \Lambda_{\leq n}$  and denote by  $\mathcal{B}_{\lambda}^K(n; \delta)$  the set of partitions  $\mu \in \Lambda_{\leq n}$  labelling cell modules in the same block of  $P_n^K(\delta)$  as  $\Delta_{\lambda}^K(n)$ . We will also say that partitions  $\mu$  and  $\lambda$  lie in the same block if they label cell modules in the same block. If the context is clear, we will write  $\mathcal{B}_{\lambda}^K(n)$  to mean  $\mathcal{B}_{\lambda}^K(n; \delta)$ .

#### 4.1. $\delta$ -pairs.

*Definition 4.2.* Let  $\lambda, \mu$  be partitions, with  $\mu \subset \lambda$ . We say that  $(\mu, \lambda)$  is a  $\delta$ -pair, written  $\mu \hookrightarrow_{\delta} \lambda$ , if  $\lambda$  differs from  $\mu$  by a strip of boxes in a single row, the last of which has content  $\delta - |\mu|$ .

*Example 4.3.* We let  $\delta = 7$ ,  $\lambda = (4, 3, 1)$  and  $\mu = (4, 1, 1)$ . Then we see that  $\lambda$  and  $\mu$  differ in precisely one row, and the last box in this row of  $\lambda$  has content 1 (see Figure 7). Since  $\delta - |\mu| = 7 - 6 = 1$ , we see that  $(\mu, \lambda)$  is a 7-pair.

Using this definition, the blocks of the partition algebra  $P_n^K(\delta)$  can be described as follows.

**Theorem 4.4** ([Mar96, Proposition 9]). *Each block of the partition algebra  $P_n^K(\delta)$  is given by a chain of partitions*

$$\lambda^{(0)} \subset \lambda^{(1)} \subset \dots \subset \lambda^{(r)}$$

where for each  $i$ ,  $(\lambda^{(i)}, \lambda^{(i+1)})$  form a  $\delta$ -pair, differing in the  $(i+1)$ -th row. Moreover there is an exact sequence of  $P_n^K(\delta)$ -modules

$$0 \rightarrow \Delta_{\lambda^{(r)}}^K(n) \rightarrow \Delta_{\lambda^{(r-1)}}^K(n) \rightarrow \dots \rightarrow \Delta_{\lambda^{(1)}}^K(n) \rightarrow \Delta_{\lambda^{(0)}}^K(n) \rightarrow L_{\lambda^{(0)}}^K(n) \rightarrow 0$$

with the image of each homomorphism a simple module. In particular, each of the cell modules  $\Delta_{\lambda^{(i)}}^K(n)$  for  $0 \leq i < r$  has Loewy structure

$$\begin{array}{c} L_{\lambda^{(i)}}^K(n) \\ L_{\lambda^{(i+1)}}^K(n) \end{array}$$

and  $\Delta_{\lambda^{(r)}}^K(n) = L_{\lambda^{(r)}}^K(n)$ .

**4.2. Reflection geometry.** We now reformulate this result in terms of the geometry of a reflection group. Let  $\{\varepsilon_0, \dots, \varepsilon_n\}$  be a set of formal symbols and set

$$E_n = \bigoplus_{i=0}^n \mathbb{R}\varepsilon_i$$

to be the  $n+1$ -dimensional space with basis  $\varepsilon_0, \dots, \varepsilon_n$ . We have an inner product  $\langle \cdot, \cdot \rangle$  on  $E_n$  given by extending linearly the relations

$$\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$$

where  $\delta_{ij}$  is the Kronecker delta.

Let  $\Phi = \{\varepsilon_i - \varepsilon_j : 0 \leq i, j \leq n\}$  be a root system of type  $A_n$ , and  $W_n \cong \mathfrak{S}_{n+1}$  the corresponding Weyl group, generated by the reflections  $s_{i,j} = s_{\varepsilon_i - \varepsilon_j}$  ( $0 \leq i < j \leq n$ ) defined by

$$s_{i,j}(x) = x - \langle x, \varepsilon_i - \varepsilon_j \rangle (\varepsilon_i - \varepsilon_j)$$

for all  $x \in E_n$ .



If we fix the element  $\rho = \rho(\delta) = (\delta, -1, -2, \dots, -n)$  we may then define a shifted action of  $W_n$  on  $E_n$ , given by

$$w \cdot_{\delta} x = w(x + \rho(\delta)) - \rho(\delta)$$

for all  $w \in W_n$  and  $x \in E_n$ .

Given a partition  $\lambda = (\lambda_1, \lambda_2, \dots) \in \Lambda_{\leq n}$ , let

$$\hat{\lambda} = (-|\lambda|, \lambda_1, \dots, \lambda_n) = -|\lambda|\varepsilon_0 + \sum_{i=1}^n \lambda_i \varepsilon_i \in E_n$$

Using this embedding of  $\Lambda_{\leq n}$  into  $E_n$  we can consider the action of  $W_n$  on the set of partitions  $\Lambda_{\leq n}$  defined by

$$w \cdot_{\delta} \hat{\lambda} = w(\hat{\lambda} + \rho(\delta)) - \rho(\delta)$$

where  $w \in W_n$  and  $\rho(\delta) = (\delta, -1, -2, \dots, -n)$ . We then have the following reformulation of [Mar96]

**Theorem 4.5.** *Let  $\lambda, \mu \in \Lambda_{\leq n}$ . Then we have  $\mu \in \mathcal{B}_{\lambda}^K(n; \delta)$  if and only if  $\hat{\mu} \in W_n \cdot_{\delta} \hat{\lambda}$ .*

*Proof.* We saw above that the blocks of  $P_n^K(\delta)$  are given by maximal chains of partitions

$$\lambda^{(0)} \subset \lambda^{(1)} \subset \dots \subset \lambda^{(r)}$$

where for each  $i$ ,  $(\lambda^{(i-1)}, \lambda^{(i)})$  form a  $\delta$ -pair, differing in the  $i$ -th row. We claim that  $\widehat{\lambda^{(i)}} = s_{0,i} \cdot_{\delta} \widehat{\lambda^{(i-1)}}$ . Indeed,

$$\begin{aligned} s_{0,i} \cdot_{\delta} \widehat{\lambda^{(i-1)}} &= (\lambda_i^{(i-1)} - i, \lambda_1^{(i-1)} - 1, \dots, -|\lambda^{(i-1)}| + \delta, \dots, \lambda_n^{(i-1)} - n) - \rho(\delta) \\ &= (\lambda_i^{(i-1)} - i - \delta, \lambda_1^{(i-1)}, \lambda_2^{(i-1)}, \dots, -|\lambda^{(i-1)}| + \delta + i, \dots, \lambda_n^{(i-1)}) \end{aligned} \quad (4.1)$$

Now the partition

$$(\lambda_1^{(i-1)}, \lambda_2^{(i-1)}, \dots, -|\lambda^{(i-1)}| + \delta + i, \dots, \lambda_n^{(i-1)})$$

obtained from (4.1) differs from  $\lambda^{(i-1)}$  by a strip of boxes in row  $i$  only, the last of which has content

$$(-|\lambda^{(i-1)}| + \delta + i) - i = \delta - |\lambda^{(i-1)}|$$

and so  $s_{0,i} \cdot_{\delta} \widehat{\lambda^{(i-1)}} = \widehat{\lambda^{(i)}}$  as claimed. Therefore if  $\mu \neq \nu \in \Lambda_{\leq n}$  are in the same block then  $\mu = \lambda^{(i)}$  and  $\nu = \lambda^{(j)}$  for some  $i < j$  say, and

$$\hat{\nu} = (s_{0,j} \dots s_{0,i+2} s_{0,i+1}) \cdot_{\delta} \hat{\mu}$$

Conversely, suppose  $\lambda, \mu \in \Lambda_{\leq n}$  satisfy  $\hat{\mu} \in W_n \cdot_{\delta} \hat{\lambda}$ . Since  $\lambda$  is a partition, the sequence  $(\lambda_1 - 1, \lambda_2 - 2, \dots, \lambda_n - n)$  is strictly decreasing, and similarly for  $\mu$ . Therefore if  $\hat{\mu} \in W_n \cdot_{\delta} \hat{\lambda}$ , then  $\hat{\mu} + \rho(\delta) = w(\hat{\lambda} + \rho(\delta))$  for some  $w \in W_n$  not fixing entry 0, and we have

$$\hat{\mu} + \rho(\delta) = (\lambda_i - i, \dots)$$

for some  $1 \leq i \leq n$ . If  $\lambda_i - i = \delta - |\lambda|$  then  $\mu = \lambda$  and the result is immediate.

If now  $\lambda_i - i < \delta - |\lambda|$ , then

$$\hat{\mu} + \rho(\delta) = (\lambda_i - i, \dots, \lambda_j - j, \delta - |\lambda|, \lambda_{j+1} - (j+1), \dots, \lambda_{i-1} - (i-1), \lambda_{i+1} - (i+1), \dots)$$

for some  $j$ . If instead  $\lambda_i - i > \delta - |\lambda|$  then

$$\begin{aligned} \hat{\mu} + \rho(\delta) &= \\ &(\lambda_i - i, \dots, \lambda_{i-1} - (i-1), \lambda_{i+1} - (i+1), \dots, \lambda_j - j, \delta - |\lambda|, \lambda_{j+1} - (j+1), \dots) \end{aligned}$$

for some  $j$ . In either case, we have

$$\hat{\mu} + \rho(\delta) = (s_{0,i} \dots s_{0,j+2} s_{0,j+1}) \cdot_{\delta} (\hat{\lambda} + \rho(\delta))$$

and using the calculation in (4.1) we see that these must be elements in a chain of  $\delta$ -pairs, and so are in the same block.  $\square$

## 5. MODULAR REPRESENTATION THEORY OF THE PARTITION ALGEBRA

In characteristic 0, the blocks of the partition algebra are determined by the orbits of the Weyl group  $W_n$  of type  $A_n$ . Our aim is to show that by replacing  $W_n$  with the corresponding affine Weyl group  $W_n^p$ , we obtain the corresponding result in the positive characteristic case. We begin by defining positive characteristic analogues of Definitions 4.1 and 4.2.

*Definition 5.1.* Let  $\lambda \in \Lambda_{\leq n}$  and denote by  $\mathcal{B}_\lambda^k(n; \delta)$  the set of partitions  $\mu \in \Lambda_{\leq n}$  labelling cell modules in the same block of  $P_n^k(\delta)$  as  $\Delta_\lambda^k(n)$ . We will also say that partitions  $\mu$  and  $\lambda$  lie in the same block if they label cell modules in the same block (equivalently if  $\mu \in \mathcal{B}_\lambda^k(n; \delta)$ ). If the context is clear, we will write  $\mathcal{B}_\lambda^k(n)$  to mean  $\mathcal{B}_\lambda^k(n; \delta)$ .

### 5.1. $(\delta, p)$ -pairs.

*Definition 5.2.* Let  $\lambda \vdash n$  and  $\mu \vdash n - t$  for some  $t \geq 0$ . We say that  $(\lambda, \mu)$  is a  $(\delta, p)$ -pair if

$$t\delta - t|\mu| - \text{ct}(\lambda) + \text{ct}(\mu) - \frac{t(t-1)}{2} = 0 \quad (5.1)$$

in the field  $k$ , where  $\text{ct}(\lambda)$  is as in (1.1).

This definition seeks to generalise the notion of  $\delta$ -pairs from Section 4.1.

In [DW00] it was shown that equation (5.1) over  $K$  provides a necessary condition for  $\lambda$  and  $\mu$  to lie in the same block of  $P_n^K(\delta)$ . We wish to extend this result to the field  $k$  of positive characteristic. Note however that the proof given in [DW00] does not generalise. Instead we will need to use the Jucys-Murphy elements introduced in [HR05]. These elements were later defined inductively in [Eny13, Section 2.3] as follows.

*Definition 5.3.* (i) Set  $L_0 = 0$ ,  $L_1 = p_1$ ,  $\sigma_1 = 1$ ,  $\sigma_2 = s_1$  and for  $i \geq 1$ , define

$$L_{i+1} = -s_i L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} L_i s_i + p_{i+\frac{1}{2}} L_i p_{i+1} p_{i+\frac{1}{2}} + s_i L_i s_i + \sigma_{i+1}$$

where for  $i \geq 2$  we define

$$\begin{aligned} \sigma_{i+1} = & s_{i-1} s_i \sigma_i s_i s_{i+1} + s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} + p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} \\ & - s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} - p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i \end{aligned}$$

(ii) Set  $L_{\frac{1}{2}} = 0$ ,  $\sigma_{\frac{1}{2}} = 1$ ,  $\sigma_{1+\frac{1}{2}} = 1$  and for  $i \geq 1$ , define

$$\begin{aligned} \sigma_{i+\frac{1}{2}} = & s_{i-1} s_i \sigma_{i-\frac{1}{2}} s_i s_{i-1} + p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i + s_i p_{i-\frac{1}{2}} L_{i-\frac{1}{2}} s_i p_{i-\frac{1}{2}} \\ & - p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} - s_i p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i \end{aligned}$$

If we project these elements onto the quotient  $P_n^k(\delta)/J_n^{(n-1)}$ , where  $J_n^{(n-1)}$  is defined in (3.1), then we obtain the following result.

**Lemma 5.4.** (i)  $\sigma_i + J_n^{(n-1)} = s_{i-1} + J_n^{(n-1)}$  for all  $i \geq 2$

(ii)  $L_i + J_n^{(n-1)} = \sum_{j=1}^{i-1} s_{j,i} + J_n^{(n-1)}$  for all  $i \geq 2$

(iii)  $\sigma_{i+\frac{1}{2}} + J_n^{(n-1)} = 1 + J_n^{(n-1)}$  for all  $i \geq 0$

(iv)  $L_{i+\frac{1}{2}} + J_n^{(n-1)} = i + J_n^{(n-1)}$  for all  $i \geq 0$

(v) Let  $Z_n = L_{\frac{1}{2}} + L_1 + L_{1+\frac{1}{2}} + \cdots + L_n$ . Then

$$Z_n + J_n^{(n-1)} = \frac{n(n-1)}{2} + \sum_{1 \leq i < j \leq n} s_{i,j} + J_n^{(n-1)}$$

*Proof.* We prove these statements by induction on  $i$ .

(i) This is true for  $i = 2$  by definition. Now let  $i \geq 2$ , then we have

$$\begin{aligned} \sigma_{i+1} + J_n^{(n-1)} &= s_{i-1} s_i \sigma_i s_i s_{i-1} + J_n^{(n-1)} \\ &= s_{i-1} s_i s_{i-1} s_i s_{i-1} + J_n^{(n-1)} \text{ by induction} \\ &= s_i + J_n^{(n-1)} \end{aligned}$$

(ii) We have  $L_2 + J_n^{(n-1)} = \sigma_2 + J_n^{(n-1)} = s_1 + J_n^{(n-1)}$ . Now let  $i \geq 2$ , then we have

$$\begin{aligned} L_{i+1} + J_n^{(n-1)} &= s_i L_i s_i + \sigma_{i+1} + J_n^{(n-1)} \\ &= s_i \left( \sum_{j=1}^{i-1} s_{j,i} \right) s_i + s_i + J_n^{(n-1)} \text{ by induction and using (i)} \\ &= \sum_{j=1}^{i-1} s_{j,i+1} + s_i + J_n^{(n-1)} \\ &= \sum_{j=1}^i s_{j,i+1} + J_n^{(n-1)} \end{aligned}$$

(iii) We have  $\sigma_{\frac{1}{2}} = 1$ , and for  $i \geq 1$

$$\begin{aligned} \sigma_{i+\frac{1}{2}} + J_n^{(n-1)} &= s_{i-1} s_i \sigma_{i-\frac{1}{2}} s_i s_{i-1} + J_n^{(n-1)} \\ &= s_{i-1} s_i 1 s_i s_{i-1} + J_n^{(n-1)} \\ &= 1 + J_n^{(n-1)} \end{aligned}$$

(iv) We have  $L_{\frac{1}{2}} = 0$ , and for  $i \geq 1$

$$\begin{aligned} L_{i+\frac{1}{2}} + J_n^{(n-1)} &= s_i L_{i-\frac{1}{2}} s_i + \sigma_{i+\frac{1}{2}} + J_n^{(n-1)} \\ &= s_i (i-1) s_i + 1 + J_n^{(n-1)} \text{ by induction and using (iii)} \\ &= i + J_n^{(n-1)} \end{aligned}$$

(v) Follows immediately from (ii) and (iv). □

Recall the following result.

**Lemma 5.5** ([HR05, Theorem 3.35], [Eny13, Lemma 3.14]). *Let  $\mu \in \Lambda_{\leq n}$  with  $|\mu| = n - t$  for some  $t \geq 0$ . Then  $Z_n$  acts on  $\Delta_{\mu}^k(n; \delta)$  as scalar multiplication by*

$$t\delta + \binom{|\mu|}{2} + \text{ct}(\mu)$$

We can now prove the following result.

**Theorem 5.6.** *Let  $\lambda, \mu \in \Lambda_{\leq n}$ . If there exists a submodule  $M \subset \Delta_{\mu}^k(n; \delta)$  with*

$$\text{Hom}(\Delta_{\lambda}^k(n; \delta), \Delta_{\mu}^k(n; \delta)/M) \neq 0.$$

*then  $(\mu, \lambda)$  is a  $(\delta, p)$ -pair.*

*Proof.* By use of the localisation functor (3.3) we may assume that  $\lambda \vdash n$  and  $\mu \vdash n - t$ . Therefore we have  $\Delta_{\lambda}^k(n; \delta) \cong S_k^{\lambda}$ , and the ideal  $J_n^{(n-1)}$  acts as zero on  $\Delta_{\lambda}^k(n; \delta)$ .

Now suppose that  $\text{Hom}(\Delta_{\lambda}^k(n; \delta), \Delta_{\mu}^k(n; \delta)/M) \neq 0$ . Then there exists a submodule  $N$  of  $\Delta_{\mu}^k(n; \delta)$  with  $M \subset N \subset \Delta_{\mu}^k(n; \delta)$  and a non-zero homomorphism

$$\Delta_{\lambda}^k(n; \delta) \cong S_k^{\lambda} \longrightarrow N/M$$

By Lemma 5.4(v), the element

$$Z_n - \frac{n(n-1)}{2} - \sum_{1 \leq i < j \leq n} s_{i,j} \quad (5.2)$$

must act as zero on  $N/M$ . We know that  $\sum_{1 \leq i < j \leq n} s_{i,j}$  acts by the scalar  $\text{ct}(\lambda)$  on  $S_k^{\lambda}$ , and hence also on  $N/M$ . Using Lemma 5.5, we then see that the element (5.2) acts on  $M/N$  by the scalar

$$\begin{aligned} & t\delta + \left( \frac{|\mu|}{2} \right) + \text{ct}(\mu) - \frac{n(n-1)}{2} - \text{ct}(\lambda) \\ &= t\delta - t|\mu| - \text{ct}(\lambda) + \text{ct}(\mu) - \frac{t(t-1)}{2} \end{aligned}$$

Since this must be zero in the field  $k$ , we see that  $\lambda$  and  $\mu$  must form a  $(\delta, p)$ -pair.  $\square$

The following theorem from [HHKP10] allows us to use the modular representation theory of the symmetric group in examining the partition algebra.

**Theorem 5.7** ([HHKP10, Corollary 6.2]). *Let  $\lambda, \mu \vdash n - t$  be partitions, with  $\lambda \in \Lambda_{\leq n}^*$ . Then*

$$[\Delta_{\mu}^k(n; \delta) : L_{\lambda}^k(n; \delta)] = [S_k^{\mu} : D_k^{\lambda}]$$

*In particular, given two partitions  $\lambda, \mu \vdash n - t$ , if the two Specht modules  $S_k^{\lambda}$  and  $S_k^{\mu}$  are in the same block for the symmetric group algebra  $k\mathfrak{S}_{n-t}$ , then  $\mu \in \mathcal{B}_{\lambda}^k(n; \delta)$ .*

**Corollary 5.8.** *If  $\delta$  does not belong to the prime subfield  $\mathbb{F}_p$  then the blocks of the partition algebra  $P_n^k(\delta)$  are given by the corresponding blocks of the symmetric groups. More precisely for each  $\lambda \in \Lambda_{\leq n}$  with  $\lambda \vdash n - t$  we have that  $\mathcal{B}_{\lambda}^k(n; \delta)$  is given by the block of  $k\mathfrak{S}_{n-t}$  containing  $\lambda$ .*

*Proof.* This follows directly from Theorem 5.6, Theorem 5.7 and the fact that the algebra  $P_n^k(\delta)$  is cellular.  $\square$

**5.2. Affine reflection geometry and a necessary condition for blocks.** As with the characteristic zero result, we now wish to reformulate this in terms of the geometry of a reflection group. Let  $W_n^p$  be the group generated by the affine reflections  $s_{i,j,rp} = s_{\varepsilon_i - \varepsilon_j, rp}$  ( $0 \leq i < j \leq n$ ),  $r \in \mathbb{Z}$ , where  $s_{i,j,rp}(x) = x - (\langle x, \varepsilon_i - \varepsilon_j \rangle - rp)(\varepsilon_i - \varepsilon_j)$ . Using the same embedding  $\hat{\lambda} \in E_n$  of a partition  $\lambda \in \Lambda_{\leq n}$  and shifted action as in Section 4.2, we have the following characterisation of the  $W_n^p$ -orbits on  $\Lambda_{\leq n}$ .

**Lemma 5.9.** *Let  $\lambda, \mu \in \Lambda_{\leq n}$ , then we have*

$$\hat{\mu} \in W_n^p \cdot_{\delta} \hat{\lambda} \iff \hat{\mu} + \rho(\delta) \sim_p \hat{\lambda} + \rho(\delta)$$

*where for  $x, y \in E_n$ ,  $x \sim_p y$  means there is a permutation  $\sigma \in \mathfrak{S}_{n+1}$  such that  $x_i \equiv y_{\sigma(i)} \pmod{p}$  for all  $0 \leq i \leq n$*

*Proof.* We have  $\hat{\mu} \in W_n^p \cdot \hat{\lambda}$  if and only if there is some  $w \in W$  and  $\alpha \in \mathbb{Z}\Phi$  such that

$$\hat{\mu} + \rho(\delta) = w(\hat{\lambda} + \rho(\delta)) + p\alpha$$

Conversely, we have  $\hat{\mu} + \rho(\delta) \sim_p \hat{\lambda} + \rho(\delta)$  if and only if

$$\hat{\mu} + \rho(\delta) = w(\hat{\lambda} + \rho(\delta)) + px$$

for some  $w \in W$  and  $x \in \mathbb{Z}^{n+1}$ . But as  $\sum_{i=0}^n (\hat{\mu})_i = \sum_{i=0}^n (\hat{\lambda})_i = 0$  we see that  $\sum_{i=0}^n x_i = 0$ , and therefore  $x \in \mathbb{Z}\Phi$ .  $\square$

**Definition 5.10.** Let  $\lambda \in \Lambda_{\leq n}$  and denote by  $\mathcal{O}_\lambda^p(n; \delta) = \{\mu \in \Lambda_{\leq n} : \hat{\mu} \in W_n^p \cdot \delta \hat{\lambda}\}$  the set of partitions in  $\Lambda_{\leq n}$  whose images in  $E_n$  are in the same  $\delta$ -shifted  $W_n^p$ -orbit as  $\hat{\lambda}$ . If the context is clear, we will write  $\mathcal{O}_\lambda^p(n)$  to mean  $\mathcal{O}_\lambda^p(n; \delta)$ .

**Theorem 5.11.** Let  $\lambda, \mu \in \Lambda_{\leq n}$ . If  $\text{Hom}(\Delta_\lambda^k(n; \delta), \Delta_\mu^k(n; \delta)/M) \neq 0$  for some  $M \subseteq \Delta_\mu^k(n; \delta)$ , then  $\mu \in \mathcal{O}_\lambda^p(n; \delta)$ .

*Proof.* By use of the localisation functor (3.3) we may assume that  $\lambda \vdash n$  and  $\mu \vdash n - t$  for some  $t \geq 0$ . We prove the result by induction on  $n$ .

If  $n = 0$  there is nothing to prove, so assume  $n \geq 1$ . If  $\lambda = \emptyset$  we must also have  $\mu = \emptyset$ , and the result holds trivially.

If now  $|\lambda| \geq 1$ , then  $\lambda$  has a removable node,  $\varepsilon_i$ , in row  $i$  say, and using (3.8) we have a surjective homomorphism

$$\text{ind}_{n-\frac{1}{2}} \Delta_{\lambda-\varepsilon_i}^k(n - \frac{1}{2}; \delta) \longrightarrow \Delta_\lambda^k(n; \delta)$$

and so by assumption and Frobenius reciprocity

$$\begin{aligned} & \text{Hom}(\text{ind}_{n-\frac{1}{2}} \Delta_{\lambda-\varepsilon_i}^k(n - \frac{1}{2}; \delta), \Delta_\mu^k(n; \delta)/M) \\ & \cong \text{Hom}(\Delta_{\lambda-\varepsilon_i}^k(n - \frac{1}{2}; \delta), \text{res}_n(\Delta_\mu^k(n; \delta)/M)) \neq 0 \end{aligned}$$

By the restriction rule (3.8) we have either

$$\text{Hom}(\Delta_{\lambda-\varepsilon_i}^k(n - \frac{1}{2}; \delta), \Delta_\mu^k(n - \frac{1}{2}; \delta)/N) \neq 0$$

for some submodule  $N \subset \Delta_\mu^k(n - \frac{1}{2}; \delta)$ , or

$$\text{Hom}(\Delta_{\lambda-\varepsilon_i}^k(n - \frac{1}{2}; \delta), \Delta_{\mu-\varepsilon_j}^k(n - \frac{1}{2}; \delta)/Q) \neq 0$$

for some removable node  $\varepsilon_j$  in row  $j$  of  $\mu$  say, and some submodule  $Q \subset \Delta_{\mu-\varepsilon_j}^k(n - \frac{1}{2}; \delta)$ .

Applying Proposition 3.1 we have the following two cases:

**Case 1:**  $\text{Hom}(\Delta_{\lambda-\varepsilon_i}^k(n - 1; \delta - 1), \Delta_\mu^k(n - 1; \delta - 1)/N) \neq 0$  for some submodule  $N \subset \Delta_\mu^k(n - 1; \delta - 1)$

**Case 2:**  $\text{Hom}(\Delta_{\lambda-\varepsilon_i}^k(n - 1; \delta - 1), \Delta_{\mu-\varepsilon_j}^k(n - 1; \delta - 1)/Q) \neq 0$  for some removable node  $\varepsilon_j$  in row  $j$  of  $\mu$ , and some submodule  $Q \subset \Delta_{\mu-\varepsilon_j}^k(n - 1; \delta - 1)$ .

**Case 1** Applying our inductive step, we have that  $\hat{\mu} \in W_n^p \cdot \widehat{\lambda - \varepsilon_i}$ . Using Lemma 5.9 we see that  $\hat{\mu} + \rho(\delta - 1) \sim_p \widehat{\lambda - \varepsilon_i} + \rho(\delta - 1)$ , that is

$$(\delta - 1 - |\mu|, \mu_1 - 1, \dots, \mu_n - n) \sim_p (\delta - |\lambda|, \lambda_1 - 1, \dots, \lambda_i - i - 1, \dots, \lambda_n - n) \quad (5.3)$$

We also see from Theorem 5.6 that  $(\lambda - \varepsilon_i, \mu)$  form a  $(\delta - 1, p)$ -pair. Since  $|\lambda - \varepsilon_i| - |\mu| = t - 1$ , we have

$$(t - 1)(\delta - 1) - (t - 1)|\mu| - \text{ct}(\lambda) + \text{ct}(\varepsilon_i) + \text{ct}(\mu) - \frac{(t - 1)(t - 2)}{2} \equiv 0 \pmod{p}$$

from which we deduce

$$t\delta - t|\mu| - \text{ct}(\lambda) + \text{ct}(\mu) - \frac{t(t-1)}{2} + \text{ct}(\varepsilon_i) + |\mu| - \delta \equiv 0 \pmod{p}$$

Moreover by assumption we have that  $(\lambda, \mu)$  is a  $(\delta, p)$ -pair, thus

$$\text{ct}(\varepsilon_i) = \lambda_i - i \equiv \delta - |\mu| \pmod{p} \quad (5.4)$$

Combining (5.3) and (5.4), the sequences

$$\hat{\lambda} + \rho(\delta) = (\delta - |\lambda|, \lambda_1 - 1, \dots, \lambda_i - i, \dots, \lambda_n - n)$$

and

$$\hat{\mu} + \rho(\delta) = (\delta - |\mu|, \mu_1 - 1, \dots, \mu_n - n)$$

are then equivalent modulo  $p$  (up to reordering). A final application of Lemma 5.9 then provides the result.

**Case 2** Applying our inductive step, we have that  $\widehat{\mu - \varepsilon_j} \in W_n^p \cdot_{\delta-1} \widehat{\lambda - \varepsilon_i}$ . Using Lemma 5.9 we see that  $\widehat{\mu - \varepsilon_j} + \rho(\delta - 1) \sim_p \widehat{\lambda - \varepsilon_i} + \rho(\delta - 1)$ , that is

$$(\delta - |\mu|, \mu_1 - 1, \dots, \mu_j - j - 1, \dots, \mu_n - n) \sim_p (\delta - |\lambda|, \lambda_1 - 1, \dots, \lambda_i - i - 1, \dots, \lambda_n - n) \quad (5.5)$$

We also see from Theorem 5.6 that  $(\lambda - \varepsilon_i, \mu - \varepsilon_j)$  form a  $(\delta - 1, p)$ -pair. Since  $|\lambda - \varepsilon_i| - |\mu - \varepsilon_j| = t$ , we have

$$t(\delta - 1) - t(|\mu| - 1) - \text{ct}(\lambda) + \text{ct}(\varepsilon_i) + \text{ct}(\mu) - \text{ct}(\varepsilon_j) - \frac{t(t-1)}{2} \equiv 0 \pmod{p}$$

Moreover by assumption we have that  $(\lambda, \mu)$  is a  $(\delta, p)$ -pair, thus

$$\text{ct}(\varepsilon_i) \equiv \text{ct}(\varepsilon_j) \pmod{p}$$

that is,

$$\lambda_i - i \equiv \mu_j - j \pmod{p} \quad (5.6)$$

Combining (5.5) and (5.6), the sequences

$$\hat{\lambda} + \rho(\delta) = (\delta - |\lambda|, \lambda_1 - 1, \dots, \lambda_i - i, \dots, \lambda_n - n)$$

and

$$\hat{\mu} + \rho(\delta) = (\delta - |\mu|, \mu_1 - 1, \dots, \mu_j - j, \dots, \mu_n - n)$$

are then equivalent modulo  $p$  (up to reordering). A final application of Lemma 5.9 then provides the result.  $\square$

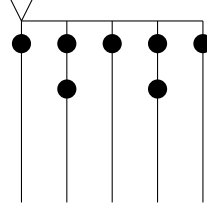
Since the blocks of  $P_n^k(\delta)$  are generated by such homomorphisms (as  $P_n^k(\delta)$  is cellular), we immediately obtain the following corollary.

**Corollary 5.12.** *Let  $\lambda, \mu \in \Lambda_{\leq n}$ . If  $\mu \in \mathcal{B}_{\lambda}^k(n; \delta)$ , then  $\mu \in \mathcal{O}_{\lambda}^p(n; \delta)$ .*

**5.3. The marked abacus and a sufficient condition for blocks.** In order to prove the converse of Corollary 5.12, i.e. that two partitions in the same  $W_n^p$ -orbit are in the same  $P_n^k(\delta)$ -block, we need to introduce a variation of the abacus of Section 1.2

Recall the result of Lemma 5.9, that  $\hat{\mu} \in W_n^p \cdot_{\delta} \hat{\lambda}$  if and only if  $\hat{\mu} + \rho(\delta) \sim_p \hat{\lambda} + \rho(\delta)$ . We represent this equivalence in the form of an abacus in the following way. For a fixed  $\delta$  and a partition  $\lambda \in \Lambda_{\leq n}$ , choose  $b \in \mathbb{N}$  satisfying  $b \geq n$ . We write  $\hat{\lambda}$  as a  $(b+1)$ -tuple by adding zeros to get a vector in  $E_b$ , and extend  $\rho(\delta)$  to the  $(b+1)$ -tuple

$$\rho(\delta) = (\delta, -1, -2, \dots, -b) \in E_b$$

FIGURE 8. The  $\delta$ -marked abacus of  $\lambda$ , with  $\lambda = (2, 1)$ ,  $p = 5$ ,  $\delta = 1$  and  $b = 7$ 

We can then define the  $\beta_\delta$ -sequence of  $\lambda$  to be

$$\begin{aligned} \beta_\delta(\lambda, b) &= \hat{\lambda} + \rho(\delta) + \underbrace{b(1, 1, \dots, 1)}_{b+1} \\ &= (\delta - |\lambda| + b, \lambda_1 - 1 + b, \lambda_2 - 2 + b, \lambda_3 - 3 + b, \dots, 2, 1, 0) \end{aligned}$$

It is clear that the equivalence in Lemma 5.9 can now also be written as  $\beta_\delta(\mu, b) \sim_p \beta_\delta(\lambda, b)$ . The  $\beta_\delta$ -sequence is used to construct the  $\delta$ -marked abacus of  $\lambda$  as follows:

- (1) Take an abacus with  $p$  runners, labelled 0 to  $p - 1$  from left to right. The positions of the abacus start at 0 and increase from left to right, moving down the runners.
- (2) Set  $v_\lambda$  to be the unique integer  $0 \leq v_\lambda \leq p - 1$  such that  $\beta_\delta(\lambda, b)_0 = \delta - |\lambda| + b \equiv v_\lambda \pmod{p}$ . Place a  $\vee$  on top of runner  $v_\lambda$ .
- (3) For the rest of the entries of  $\beta_\delta(\lambda, b)$ , place a bead in the corresponding position of the abacus, so that the final abacus contains  $b$  beads, as in Section 1.2.

Example 5.13 below demonstrates this construction.

*Example 5.13.* Let  $p = 5$ ,  $\delta = 1$ ,  $\lambda = (2, 1)$ . We choose an integer  $b \geq 3$ , for instance  $b = 7$ . Then the  $\beta$ -sequence is

$$\begin{aligned} \beta_\delta(\lambda, 7) &= (1 - 3 + 7, 2 - 1 + 7, \dots, 0) \\ &= (5, 8, 6, 4, 3, 2, 1, 0) \end{aligned}$$

The resulting  $\delta$ -marked abacus is given in Figure 8.

Note that if we ignore the  $\vee$  we recover James' abacus representing  $\lambda$  with  $b$  beads explained in Section 1.2.

If the context is clear, we will use *marked abacus* to mean  $\delta$ -marked abacus.

Recall the definition of  $\Gamma(\lambda, b)$  from (1.2). If we now use the marked abacus, we similarly define  $\Gamma_\delta(\lambda, b) = (\Gamma_\delta(\lambda, b)_0, \Gamma_\delta(\lambda, b)_1, \dots, \Gamma_\delta(\lambda, b)_{p-1})$  by

$$\Gamma_\delta(\lambda, b)_i = \begin{cases} \Gamma(\lambda, b)_i & \text{if } i \neq v_\lambda \\ \Gamma(\lambda, b)_i + 1 & \text{if } i = v_\lambda \end{cases}$$

This provides us with a further form of Lemma 5.9:

$$\begin{aligned} \hat{\mu} \in W_n^p \cdot_\delta \hat{\lambda} &\iff \hat{\mu} + \rho(\delta) \sim_p \hat{\lambda} + \rho(\delta) \\ &\iff \beta_\delta(\mu, b) \sim_p \beta_\delta(\lambda, b) \\ &\iff \Gamma_\delta(\mu, b) = \Gamma_\delta(\lambda, b) \end{aligned} \tag{5.7}$$

This gives a characterisation of the orbits of  $W_n^p$  in terms of the beads on the marked abacus.

We now use the  $\delta$ -marked abacus to show that each  $W_n^p$ -orbit contains a unique minimal element.

*Definition 5.14.* Let  $\lambda \in \Lambda_{\leq n}$ . For  $\mathcal{O} = \mathcal{O}_\lambda^p(n; \delta)$  we define  $\lambda_{\mathcal{O}}$  to be the partition such that

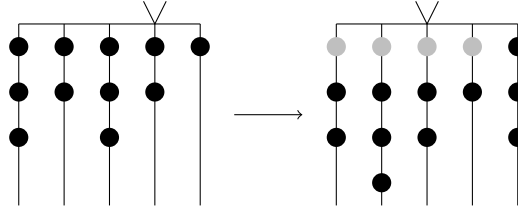


FIGURE 9. Adding 4 beads (coloured grey) to the abacus of  $\lambda_{\mathcal{O}} = (2, 1)$ . Each existing bead (coloured black) and the  $\vee$  is moved 4 places to the right.

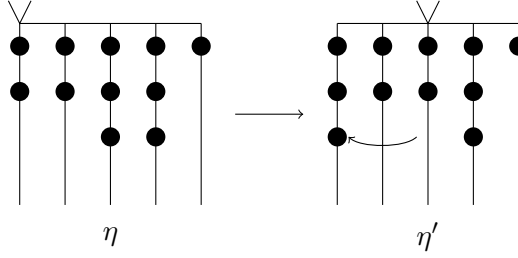


FIGURE 10. Constructing the marked abacus of  $\eta'$  from  $\eta$  in Case (2)

- (i)  $\Gamma_{\delta}(\lambda_{\mathcal{O}}, b) = \Gamma_{\delta}(\lambda, b)$ ,
- (ii) All beads on the marked abacus of  $\lambda_{\mathcal{O}}$  are as far up their runners as possible,
- (iii) The runner  $v_{\lambda_{\mathcal{O}}}$  is the rightmost runner  $i$  such that  $\Gamma_{\delta}(\lambda_{\mathcal{O}}, b)_i$  is maximal.

The partition  $\lambda_{\mathcal{O}}$  is well defined, i.e. it is independent of the number of beads used. To see this, note that by adding  $m$  beads to the abacus we move each existing bead  $m$  places to the right. Moreover since  $v_{\lambda_{\mathcal{O}}} \equiv \delta - |\lambda_{\mathcal{O}}| + b \pmod{p}$ , we also move the  $\vee$  by  $m$  places to the right. Therefore none of the beads change their relative positions to one another and the partition  $\lambda_{\mathcal{O}}$  remains unchanged. This is illustrated in Figure 9.

**Proposition 5.15.** *Let  $\lambda \in \Lambda_{\leq n}$ , then the orbit  $\mathcal{O} = \mathcal{O}_{\lambda}^p(n)$  contains a unique element of minimal degree, namely  $\lambda_{\mathcal{O}}$ . More precisely, if  $\mu \in \mathcal{O}$  then  $|\mu| \geq |\lambda_{\mathcal{O}}|$ , with equality if and only if  $\mu = \lambda_{\mathcal{O}}$ .*

*Proof.* Let  $\mu \in \mathcal{O}_{\lambda}^p(n)$  with  $\mu \neq \lambda_{\mathcal{O}}$ . Since  $\Gamma_{\delta}(\mu, b) = \Gamma_{\delta}(\lambda, b)$ , it's easy to see that there is a sequence of partition in  $\mathcal{O}$

$$\mu = \eta^{(0)}, \eta^{(1)}, \dots, \eta^{(t)} = \lambda_{\mathcal{O}}$$

for some  $t > 0$  such that for each  $0 \leq i \leq t - 1$  the partitions  $\eta = \eta^{(i)}$  and  $\eta' = \eta^{(i+1)}$  are related in precisely one of the following ways.

**Case 1** The partition  $\eta$  is not a  $p$ -core and the marked abacus of  $\eta'$  is obtained from the marked abacus of  $\eta$  by pushing a bead one step up its runner. In this case we have  $|\eta'| = |\eta| - p$  and so  $v_{\eta'} = v_{\eta}$  and  $\eta' \in \mathcal{O}_{\eta}^k(n)$ .

**Case 2** The partition  $\eta$  is a  $p$ -core. Then as  $\eta \neq \lambda_{\mathcal{O}}$ , it does not satisfy condition (iii) above. Now pick the first runner, say runner  $j$ , to the right of  $v_{\eta}$  satisfying  $\Gamma_{\delta}(\eta, b)_{v_{\eta}} \leq \Gamma_{\delta}(\eta, b)_j$ . Then the marked abacus of  $\eta'$  is obtained from that of  $\eta$  by moving the lowest bead on runner  $j$  exactly  $j - v_{\eta}$  steps to the left to runner  $v_{\eta}$ . In this case we have  $|\eta'| = |\eta| - (j - v_{\eta})$  and so we have  $v_{\eta'} = j$  and  $\eta' \in \mathcal{O}_{\eta}^k(n)$ . This is illustrated in Figure 10.

In both cases we saw that  $|\eta'| < |\eta|$  and so we get  $|\lambda_{\mathcal{O}}| < \mu$  as required.  $\square$

The aim now is to show that any partition is in the same block as the minimal element in its orbit. This will imply that the orbits do indeed coincide with the blocks. We first need the



following proposition to relate blocks over a field of characteristic zero to those over a field of positive characteristic.

**Proposition 5.16.** *If  $\mu \in \mathcal{B}_\lambda^K(n; \delta + rp)$  for some  $r \in \mathbb{Z}$ , then  $\mu \in \mathcal{B}_\lambda^k(n; \delta)$ .*

*Proof.* By the cellularity of  $P_n^K(\delta)$ , partitions  $\lambda$  and  $\mu$  are in the same  $K$ -block if and only if there is a sequence of partitions

$$\lambda = \lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(t)} = \mu$$

and  $P_n^K(\delta)$ -modules

$$M^{(i)} \leq \Delta_{\lambda^{(i)}}^K(n; \delta + rp) \quad (1 < i \leq t)$$

such that for each  $1 \leq i < t$

$$\text{Hom}(\Delta_{\lambda^{(i)}}^K(n; \delta + rp), \Delta_{\lambda^{(i+1)}}^K(n; \delta + rp)/M^{(i+1)}) \neq 0$$

Since  $\delta + rp = \delta$  in  $k$ , and the cell modules have  $R$ -forms  $\Delta_{\lambda^{(i)}}^R(n)$  such that  $k \otimes_R \Delta_{\lambda^{(i)}}^R(n) = \Delta_{\lambda^{(i)}}^k(n)$ , the same sequence of partitions link  $\lambda$  and  $\mu$  in  $\mathcal{B}_\lambda^k(n, \delta)$ . □

We now set  $b = n$ , so that all marked abaci have  $n$  beads.

**Proposition 5.17.** *Let  $\lambda \in \Lambda_{\leq n}$ . If  $\lambda \neq \lambda_\mathcal{O}$ , i.e.  $\lambda$  is not minimal in its orbit, then there exists a partition  $\mu \in \mathcal{O}_\lambda^p(n)$  with  $|\mu| < |\lambda|$  and  $\mu \in \mathcal{B}_\lambda^k(n)$ .*

*Proof.* If  $\lambda \neq \lambda_\mathcal{O}$ , then as in the proof of Proposition 5.15 either  $\lambda$  is not a  $p$ -core or  $v_\lambda$  is not the rightmost runner  $i$  such that  $\Gamma_\delta(\lambda, n)_i$  is maximal. We now refine the cases provided in the proof of Proposition 5.15 to construct a partition  $\mu$  with the required properties.

**Case A** The partition  $\lambda$  is not a  $p$ -core and there is a bead, say the  $j$ -th bead, which lies on runner  $v_\lambda$  and has an empty space immediately above it. Let  $\mu$  be the partition obtained by moving the  $j$ -th bead one space up its runner (as illustrated in Figure 11), so that it now occupies position  $\lambda_j - j + n - p$ . Note that  $|\mu| = |\lambda| - p$ . In particular, since no beads are changing runners we get  $\Gamma_\delta(\mu, n) = \Gamma_\delta(\lambda, n)$  and so  $\mu \in \mathcal{O}_\lambda^p(n)$ . Note that setwise the  $\beta_\delta$ -sequence  $\beta_\delta(\mu, n)$  must be

$$(\delta - |\lambda| + p + n, \lambda_1 - 1 + n, \dots, \lambda_j - j - p + n, \dots, 0) \quad (5.8)$$

since no other beads move. However since bead  $j$  lies on runner  $v_\lambda$ , we can find  $r \in \mathbb{Z}$  such that

$$\delta - |\lambda| + n + (r + 1)p = \lambda_j - j + n$$

and can therefore rewrite (5.8) as

$$(\lambda_j - j + n - rp, \lambda_1 - 1 + n, \dots, \delta - |\lambda| + n + rp, \dots, 0)$$

Thus, for an appropriate element  $w \in \langle s_{i,j} : 1 \leq i < j \leq n \rangle \cong \mathfrak{S}_n$ , we have:

$$w^{-1}(\beta_\delta(\mu, n)) = (\lambda_j - j - rp + n, \lambda_1 - 1 + n, \dots, \underbrace{\delta - |\lambda| + rp + n}_{j\text{-th place}}, \dots, 0)$$

and hence

$$\begin{aligned} \beta_\delta(\lambda, n) - w^{-1}(\beta_\delta(\mu, n)) &= (\delta - |\lambda| - \lambda_j + j + rp)(\varepsilon_0 - \varepsilon_j) \\ &= \langle \hat{\lambda} + \rho(\delta + rp), \varepsilon_0 - \varepsilon_j \rangle (\varepsilon_0 - \varepsilon_j) \end{aligned}$$

We can rewrite this as

$$w^{-1}(\hat{\mu} + \rho(\delta) + n(1, \dots, 1)) = \hat{\lambda} + \rho(\delta) + n(1, \dots, 1) - \langle \hat{\lambda} + \rho(\delta + rp), \varepsilon_0 - \varepsilon_j \rangle (\varepsilon_0 - \varepsilon_j)$$

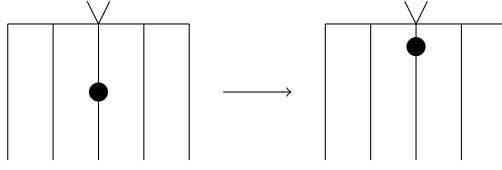


FIGURE 11. The movement of beads in Case A

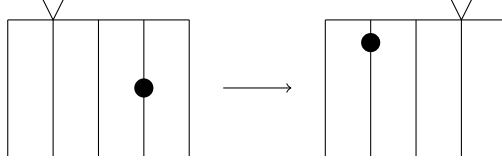


FIGURE 12. The movement of beads in Case B

Since  $\mathfrak{S}_n$  does not act on the 0-th postition, both the elements  $(rp, 0, 0, \dots, 0)$  and  $n(1, 1, \dots, 1)$  are unchanged by  $w^{-1}$ . Thus:

$$\begin{aligned}
 & w^{-1}(\hat{\mu} + \rho(\delta) + (rp, 0, \dots, 0)) \\
 &= \hat{\lambda} + \rho(\delta) + (rp, 0, \dots, 0) - \langle \hat{\lambda} + \rho(\delta + rp), \varepsilon_0 - \varepsilon_j \rangle (\varepsilon_0 - \varepsilon_j) \\
 \implies & w^{-1}(\hat{\mu} + \rho(\delta + rp)) = s_{0,j}(\hat{\lambda} + \rho(\delta + rp)) \\
 \implies & \hat{\mu} = ws_{0,j}(\hat{\lambda} + \rho(\delta + rp)) - \rho(\delta + rp) \\
 \implies & \hat{\mu} = ws_{0,j} \cdot_{\delta+rp} \hat{\lambda}
 \end{aligned}$$

Therefore  $\hat{\mu} \in W_n \cdot_{\delta+rp} \hat{\lambda}$ , and so by Theorem 4.5  $\mu \in \mathcal{B}_\lambda^K(n; \delta + rp)$ . Proposition 5.16 then provides the final result.

**Case B** The partition  $\lambda$  is not a  $p$ -core and the runner  $v_\lambda$  is empty. As  $\lambda$  is not a  $p$ -core, there is a bead, say the  $j$ -th bead (not on the runner  $v_\lambda$ ) with an empty space immediately above it. Then by moving the  $j$ -th bead one space up its runner and then across to runner  $v_\lambda$ , we obtain the abacus of a new partition  $\mu$  with  $|\mu| = |\lambda| - m$  for some  $m > 0$  (as illustrated in Figure 12). Since bead  $j$  is now on runner  $v_\lambda$  and occupies position  $\lambda_j - j + n - m$ , we see that

$$\lambda_j - j + n - m = \delta - |\lambda| + n + rp \quad (5.9)$$

for some  $r \in \mathbb{Z}$ . The runner  $v_\mu$  is given by

$$\begin{aligned}
 \delta - |\mu| + n &= \delta - |\lambda| + m + n \\
 &= \lambda_j - j + n - rp
 \end{aligned}$$

So runner  $v_\mu$  is equal to the runner previously occupied by bead  $j$ . Therefore  $\Gamma_\delta(\mu, n) = \Gamma_\delta(\lambda, n)$ , and so  $\mu \in \mathcal{O}_\lambda^p(n)$ . We also have that setwise the  $\beta_\delta$ -sequence  $\beta_\delta(\mu, n)$  is

$$(\delta - |\lambda| + m + n, \lambda_1 - 1 + n, \dots, \lambda_j - j - m + n, \dots, 0)$$

which, by using (5.9), we may rewrite as

$$(\lambda_j - j + n - rp, \lambda_1 - 1 + n, \dots, \delta - |\lambda| + n + rp, \dots, 0).$$

We can then continue as in Case A to deduce that  $\mu \in \mathcal{B}_\lambda^k(n; \delta)$ .

**Case C** The partition  $\lambda$  is not a  $p$ -core, there is at least one bead on runner  $v_\lambda$  and all the beads on runner  $v_\lambda$  are as far up as possible. Then there is a bead, say the  $j$ -th bead, not on the runner  $v_\lambda$ , with an empty space immediately above it. Define  $\nu$  to be the partition obtained by moving the  $j$ -th bead one space up and moving the last bead on runner  $v_\lambda$  one space down. By Theorems 2.3

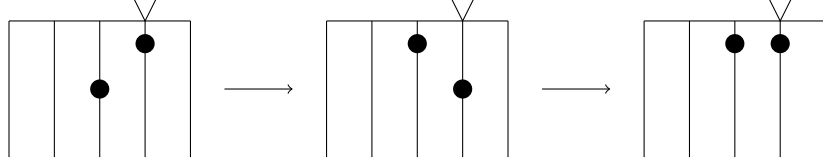


FIGURE 13. The movement of beads in Case C

and 5.7 we have that  $\nu \in \mathcal{B}_\lambda^k(n; \delta)$ . Now note that  $|\nu| = |\lambda|$  and  $\nu$  satisfies the conditions of Case A, and using this case we can find  $\mu \in \mathcal{B}_\nu^k(n; \delta) = \mathcal{B}_\lambda^k(n; \delta)$  with  $|\mu| < |\nu| = |\lambda|$ . This is illustrated in Figure 13.

**Case D** Now suppose that  $\lambda$  is a  $p$ -core, but  $v_\lambda$  is not the rightmost runner  $i$  such that  $\Gamma_\delta(\lambda, n)_i$  is maximal. As in Case 2 in the proof of Proposition 5.15 we can pick the first runner, say runner  $i$ , to the right of  $v_\lambda$  satisfying  $\Gamma_\delta(\lambda, b)_{v_\lambda} \leq \Gamma_\delta(\lambda, b)_i$  and define  $\mu$  to be the partition obtained by moving the lowest bead on runner  $i$ , say it is the  $j$ -th bead, exactly  $i - v_\lambda$  steps to the left to runner  $v_\lambda$  (as illustrated in Figure 10). Then we have  $|\mu| = |\lambda| - (i - v_\lambda)$  and so  $v_\mu = i$ . Now we have

$$\hat{\mu} = (-|\lambda| + (i - v_\lambda), \lambda_1, \dots, \lambda_{j-1}, \lambda_j - (i - v_\lambda), \lambda_{j+1}, \dots, 0)$$

As  $\lambda_j - j + n \equiv i \pmod{p}$  and  $-|\lambda| + \delta + n \equiv v_\lambda \pmod{p}$  we get that

$$\lambda_j - j - i = -|\lambda| + \delta - v_\lambda + rp$$

for some  $r \in \mathbb{Z}$ . This gives  $-|\lambda| + (i - v_\lambda) = \lambda_j - j - (\delta + rp)$  and  $\lambda_j - (i - v_\lambda) = -|\lambda| + (\delta + rp) + j$ . Thus we have

$$\begin{aligned} \hat{\mu} &= (\lambda_j - j - (\delta + rp), \lambda_1, \dots, \lambda_{j-1}, -|\lambda| + (\delta + rp) + j, \lambda_{j+1}, \dots, 0) \\ &= s_{o,j} \cdot \delta + rp \hat{\lambda}. \end{aligned}$$

Using Theorem 4.5 and Proposition 5.16 we deduce that  $\mu \in \mathcal{B}_\lambda^k(n; \delta)$ .  $\square$

We immediately deduce the following:

**Corollary 5.18.** *Let  $\lambda, \mu \in \Lambda_{\leq n}$ . If  $\mu \in \mathcal{O}_\lambda^p(n)$ , then  $\mu \in \mathcal{B}_\lambda^k(n)$ .*

*Proof.* Since each orbit  $\mathcal{O} = \mathcal{O}_\lambda^p(n)$  contains a unique minimal element  $\lambda_{\mathcal{O}}$ , it suffices to show that  $\lambda_{\mathcal{O}} \in \mathcal{B}_\lambda^k(n)$ . We prove this by induction on  $|\lambda|$ .

If  $|\lambda|$  is minimal, then  $\lambda = \lambda_{\mathcal{O}}$  and there is nothing to prove. So suppose otherwise, i.e. that  $\lambda \neq \lambda_{\mathcal{O}}$ . Then by Proposition 5.17 there is a partition  $\nu \in \mathcal{O}_\lambda^p(n)$  with  $|\nu| < |\lambda|$  and  $\nu \in \mathcal{B}_\lambda^k(n)$ . By our inductive step, we then have  $\lambda_{\mathcal{O}} \in \mathcal{B}_\nu^k(n)$ . But blocks are either disjoint or coincide entirely, so  $\mathcal{B}_\nu^k(n) = \mathcal{B}_\lambda^k(n)$  and  $\lambda_{\mathcal{O}} \in \mathcal{B}_\lambda^k(n)$  as required.  $\square$

We can now combine Corollaries 5.12 and 5.18 to obtain the main result of this section.

**Theorem 5.19.** *Let  $\lambda \in \Lambda_{\leq n}$ , then  $\mathcal{B}_\lambda^k(n; \delta) = \mathcal{O}_\lambda^p(n; \delta)$ . In other words, two partitions label cell modules in the same  $P_n^k(\delta)$ -block if and only if their images in  $E_n$  are in the same  $W_n^p$  orbit under the  $\delta$ -shifted action.*

**5.4. Limiting blocks.** The proof of Proposition 5.17 uses two main ingredients to show that a partition  $\lambda$  is in the same block as the minimal element in its orbit  $\lambda_{\mathcal{O}}$ , namely

- (1) the blocks of the partition algebra  $P_n^K(\delta + rp)$  for all  $r \in \mathbb{Z}$  (used in Cases A, B and D), and
- (2) the blocks of the symmetric group algebras  $k\mathfrak{S}_{n-t}$  for all  $0 \leq t \leq n$  (used in Case C).

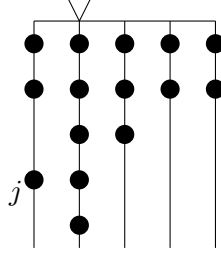


FIGURE 14. The marked abacus of  $\lambda = (7, 3^2, 1^2)$  with  $p = 5$ ,  $\delta = 1$  and  $b = 15$

One might wonder whether we can give a proof of this result which only uses (1). This would give a proof of the modular blocks of the partition algebra without assuming any result about the modular representation theory of the symmetric group (note that Sections 5.1 and 5.2 giving the necessary condition for the blocks do not use the modular representation theory of the symmetric group). This is not possible as the following example shows.

*Example 5.20.* Consider the partition  $\lambda = (7, 3^2, 1^2)$  with  $p = 5$ ,  $n = 15$  and  $\delta = 1$ . Then the marked abacus of  $\lambda$  with 15 beads is illustrated in Figure 14.

From the abacus we see that the  $W_n^p$ -orbit containing  $\lambda$  is given by

$$\mathcal{O}_\lambda^5(15) = \{(12, 3), (7, 4^2), (7, 3^2, 1^1), (7, 3, 2, 1^3), (7, 3, 1^5), (7, 3)\}.$$

Now the only  $\delta + rp$ -pair, for any  $r \in \mathbb{Z}$ , among these partitions is given by

$$(7, 3) \hookrightarrow_{21} (12, 3).$$

Thus in this case it is impossible to show that  $\lambda$  is in the same block as  $\lambda_{\mathcal{O}} = (7, 3)$  without using the blocks of the symmetric group algebra  $k\mathfrak{S}_{15}$ .

However, we will show that if we allow ourselves to increase the size of the partitions we consider, then it is possible to link any partition  $\lambda$  to the minimal element  $\lambda_{\mathcal{O}}$  of its orbit using only (the modular reduction of) the blocks of the partition algebras  $P_n(\delta + rp)$  (for all  $r \in \mathbb{Z}$ ). We make this more precise by defining the notion of *limiting blocks*.

Recall that the globalisation functor  $G_n$  defined in Section 3.1 gives a full embedding of the category  $P_n^k(\delta)\text{-mod}$  into  $P_n^k(\delta)\text{-mod}$ . Moreover, we have  $G_n(\Delta_\lambda^k(n)) = \Delta_\lambda^k(n+1)$ . So under this embedding the labelling sets for cell modules correspond via the natural inclusion  $\Lambda_{\leq n} \subset \Lambda_{\leq n+1}$ . We define

$$\Lambda = \bigcup_{n \geq 0} \Lambda_{\leq n}.$$

Now for  $\lambda \in \Lambda_{\leq n}$  the functor  $G_n$  gives an embedding  $\mathcal{B}_\lambda^k(n) \subseteq \mathcal{B}_\lambda^k(n+1)$  and so we can define the *limiting block* containing  $\lambda \in \Lambda_{\leq n}$  by

$$\mathcal{B}_\lambda^k := \bigcup_{m \geq n} \mathcal{B}_\lambda^k(m) = \{\mu \in \Lambda : \mu \in \mathcal{B}_\lambda^k(m) \text{ for some } m\}.$$

Now we consider  $E_n \subset E_{n+1}$  by taking the last coordinate to be zero and so we get  $W_n^p \subset W_{n+1}^p$ . Then we can define

$$E = \bigoplus_{i \geq 0} \mathbb{R}\epsilon_i \quad \text{and} \quad W^p = \bigcup_{n \geq 1} W_n^p,$$

and for  $\lambda \in \Lambda_{\leq n}$  we have the infinite orbit

$$\mathcal{O}_\lambda^p = \bigcup_{m \geq n} \mathcal{O}_\lambda^p(m) = \{\mu \in \Lambda : \hat{\mu} \in W^p \cdot_\delta \hat{\lambda}\}.$$

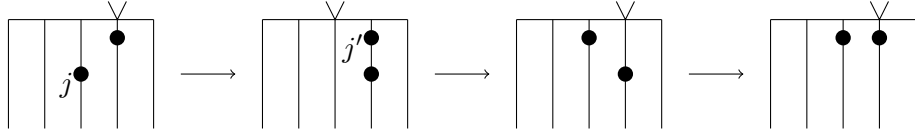


FIGURE 15. The movement of beads in Case C'

With this notion of the *limiting block* of the partition algebra, we prove the following without using any results on the modular representation theory of  $\mathfrak{S}_n$ .

**Theorem 5.21.** *Let  $\lambda \in \Lambda_{\leq n}$ , then  $\mathcal{B}_\lambda^k = \mathcal{O}_\lambda^k$ . In other words, two partitions label cell modules in the same  $P_n^k(\delta)$ -block for some  $n$  if and only if their images in  $E$  are in the same  $W^p$ -orbit under the  $\delta$ -shifted action.*

*Proof.* Suppose  $\mu \in \mathcal{B}_\lambda^k$ . Then  $\mu \in \mathcal{B}_\lambda^k(n)$  for some  $n \in \mathbb{N}$ , and so by Corollary 5.12 we have  $\mu \in \mathcal{O}_\lambda^p(n) \subset \mathcal{O}_\lambda^p$ . Suppose now that  $\mu \in \mathcal{O}_\lambda^p$ . Again we have  $\mu \in \mathcal{O}_\lambda^p(n)$  for some  $n \in \mathbb{N}$ . We can follow the proof of Proposition 5.17 but replace Case C with the following alternative:

**Case C'** The partition  $\lambda$  is not a  $p$ -core, the runner  $v_\lambda$  is not empty and all the beads on runner  $v_\lambda$  are as far up as possible. Then there is a bead, say the  $j$ -th bead, which does not lie on runner  $v_\lambda$  with a space immediately above it. Now consider the partition  $\mu$  obtained by moving bead  $j$  into the first empty space of runner  $v_\lambda$ . We then have  $|\mu| = |\lambda| + m$  for some  $m \in \mathbb{Z}$ , and as in (5.9) we have

$$\lambda_j - j + n + m = \delta - |\lambda| + n + rp \quad (5.10)$$

for some  $r \in \mathbb{Z}$ . The runner  $v_\mu$  is given by

$$\begin{aligned} \delta - |\mu| + n &= \delta - |\lambda| - m + n \\ &= \lambda_j - j + n - rp \end{aligned}$$

So runner  $v_\mu$  is equal to the runner previously occupied by bead  $j$  (see Figure 15). Therefore  $\Gamma_\delta(\mu, n) = \Gamma_\delta(\lambda, n)$ , but if  $m > 0$  then  $|\mu| > |\lambda|$  so we may not have  $\mu \in \mathcal{O}_\lambda^p(n)$ . However it is true that  $\hat{\mu} \in W^p \cdot_\delta \hat{\lambda}$ , so  $\mu \in \mathcal{O}_\lambda^k$ . We also have that setwise the  $\beta_\delta$ -sequence  $\beta_\delta(\mu, n)$  is

$$(\delta - |\lambda| - m + n, \lambda_1 - 1 + n, \dots, \lambda_j - j + m + n, \dots, 0)$$

which, by using (5.10), we may rewrite as

$$(\lambda_j - j + n - rp, \lambda_1 - 1 + n, \dots, \delta - |\lambda| + n + rp, \dots, 0)$$

Arguing as in Case A of Proposition 5.17 we see that there is some  $w \in \mathfrak{S}_n$  such that

$$\hat{\mu} = ws_{0,j} \cdot_{\delta-rp} \hat{\lambda}$$

and so by Theorem 4.5 and Proposition 5.16,  $\mu \in \mathcal{B}_\lambda^k$ .

Now that bead  $j$  occupies the lowest position on runner  $v_\lambda$ , let bead  $j'$  be the bead immediately above this. Let  $\nu$  be the partition obtained by moving bead  $j'$  into the space above the position previously occupied by bead  $j$ , i.e. into position  $\lambda_j - j + n - p$ . Then  $|\nu| = |\mu| + m$  for some  $m \in \mathbb{Z}$ , and we repeat the argument of the previous paragraph to see that  $\nu \in \mathcal{B}_\mu^k = \mathcal{B}_\lambda^k$ .

We repeat this process for each bead that falls into Case C of Proposition 5.17. Then we are left only with beads in Case A, B or D, and may therefore continue with the proof of Proposition 5.17 to get the result.  $\square$

## REFERENCES

- [DW00] W. F. Doran and D. B. Wales, *The Partition Algebra Revisited*, J. Algebra **231** (2000), 265–330.
- [Eny13] J. Enyang, *A seminormal form for partition algebras*, J. Comb. Theory Ser. A **120** (2013), 1737–1785.
- [GL96] J. J. Graham and G. I. Lehrer, *Cellular algebras*, Invent. Math. **123** (1996), 1–34.
- [Gre80] J. A. Green, *Polynomial representations of  $GL_n$* , Lecture Notes in Mathematics, vol. 830, Springer, 1980.
- [HHKP10] R. Hartmann, A. Henke, S. König, and R. Paget, *Cohomological stratification of diagram algebras*, Math. Ann. **347** (2010), 765–804.
- [HR05] T. Halverson and A. Ram, *Partition algebras*, European J. Combin. **26** (2005), 869–921.
- [Jam78] G. D. James, *The representation theory of the symmetric groups*, Lecture Notes in Mathematics, vol. 682, Springer-Verlag, 1978.
- [JK81] G. D. James and A. Kerber, *The representation theory of the symmetric group*, Encyclopedia of Mathematics and its Applications, vol. 16, Addison-Wesley, 1981.
- [Mar94] P. P. Martin, *Temperley-lieb algebras for non-planar statistical mechanics – the partition algebra construction*, J. Knot Theory Ramifications **3** (1994), 51–82.
- [Mar96] ———, *The Structure of the Partition Algebras*, J. Algebra **183** (1996), 319–358.
- [Mar00] ———, *The partition algebra and the potts model transfer matrix spectrum in high dimensions*, J. Phys. A **33** (2000), no. 19, 3669–3695.
- [Xi99] C. Xi, *Partition algebras are cellular*, Compositio Mathematica **119** (1999), 107–118.

*E-mail address:* `chris.bowman.2@city.ac.uk`

*E-mail address:* `Maud.Devisscher.1@city.ac.uk`

*E-mail address:* `Oliver.King.1@city.ac.uk`

DEPARTMENT OF MATHEMATICS, CITY UNIVERSITY LONDON, NORTHAMPTON SQUARE, LONDON, EC1V 0HB, ENGLAND.